

LIMITS

1.

Right hand and Left hand Limit

To evaluate $\lim_{x \rightarrow a^+} f(x)$

- Put $x = a + h$ in $f(x)$ to get $\lim_{h \rightarrow 0} f(a+h)$
- Take the limit as $h \rightarrow 0$.

To evaluate $\lim_{x \rightarrow a^-} f(x)$.

- Put $x = a - h$ in $f(x)$ to get $\lim_{h \rightarrow 0} f(a-h)$.
- Take the limit as $h \rightarrow 0$.

2.

Some Useful Limits

(1) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$, where $n \in \mathbb{Q}$, the set of rational numbers.

(2)

$$(i) \lim_{n \rightarrow \infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } -1 < a < 1 \\ \text{does not exist,} & \text{if } a \leq -1 \end{cases}$$

$$(ii) \lim_{x \rightarrow \infty} \frac{a_0 x^p + a_1 x^{p-1} + \dots + a_{p-1} x + a_p}{b_0 x^q + b_1 x^{q-1} + \dots + b_{q-1} x + b_q}$$

$$= \begin{cases} \frac{a_0}{b_0}, & \text{if } p = q \\ 0, & \text{if } p < q \\ \infty, \frac{a_0}{b_0} > 0 & \text{if } p > q \end{cases}$$

(3) Trigonometric Limits

(i) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x \rightarrow 0} \frac{\cos x}{x} = 1$

(iii) $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

(iv) $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1$

(v) $\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$

(vi) $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$.

Some Useful Expansions

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ to ∞

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ to ∞

(iii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \dots$ to ∞

(iv) $\sin^{-1} x = x + \frac{1^2 \cdot x^3}{3!} + \frac{1^2 \cdot 3^2}{5!} x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!} x^7 + \dots$ to ∞

(v) $(\sin^{-1} x)^2 = \frac{2}{2!} x^2 + \frac{2 \cdot 2^2}{4!} x^4 + \frac{2 \cdot 2^2 \cdot 4^2}{6!} x^6 + \dots$ to ∞

(vi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ to ∞ .

(4) Exponential and Logarithmic Limits

(i) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

(ii) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$, $a > 0$

(iii) $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \log_e \left(\frac{a}{b} \right)$; $a, b > 0$

(iv) $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = n$

(v) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$

(vi) $\lim_{h \rightarrow 0} (1+ah)^{1/h} = e^a$

(vii) $\lim_{x \rightarrow \infty} \frac{\log x}{x^m} = 0$, ($m > 0$)

(viii) $\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e$, ($a > 0, a \neq 1$)

(ix) $\lim_{x \rightarrow 0} \left(1 + \frac{a}{x} \right)^x = e^a$

(x) $\lim_{x \rightarrow 0} \left(1 + \frac{1}{f(x)} \right)^{f(x)} = e$, where $f(x) \rightarrow \infty$ as $x \rightarrow 0$.

(xi) $\lim_{x \rightarrow 0} (1+f(x))^{1/f(x)} = e$, where $f(x) \rightarrow 0$ as $x \rightarrow 0$

Some Useful Expansions

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ to ∞

(ii) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$ to ∞

$$(iii) \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \text{ to } \infty, -1 < x \leq 1$$

$$(iv) \log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \text{ to } \infty, -1 \leq x < 1$$

$$(v) a^x = e^{x \log a} = 1 + x \log a + \frac{(x \log a)^2}{2!} + \dots \text{ to } \infty.$$

$$(vi) (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots \text{ to } \infty, -1 < x < 1,$$

Note

(i) If $\lim_{x \rightarrow a} f(x) = A > 0$ and $\lim_{x \rightarrow a} g(x) = B$, then $\lim_{x \rightarrow a} [f(x)]^{g(x)} = A^B$.

(ii) If $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow 0} g(x) = \infty$, then

$$\lim_{x \rightarrow a} [f(x)]^{g(x)} = e^{\lim_{x \rightarrow a} g(x)[f(x)-1]}.$$

3.

EVALUATION OF LIMITS USING L'HOSPITAL'S RULE

(i) $\left(\frac{0}{0}\right)$ form: If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} =$

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \text{ provided the limit on the R.H.S. exists.}$$

(ii) $\left(\frac{\infty}{\infty}\right)$ form: If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$

$$= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \text{ provided the limit on the R.H.S. exists.}$$

Note: That sometimes we have to repeat the process if the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again.